Problem Set 7
Statistical Mechanics
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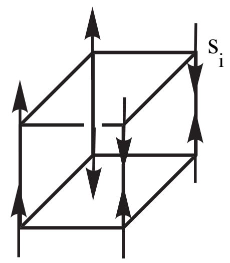
Problem 18: Model ferromagnet revisited

This problem focusses on the same model ferromagnet we had already encountered in problem 14 - but here our approach is different.

We consider the cubic lattice of N sites shown in the graph. Each site is occupied by an arrow pointing up (+1) or down (-1).



The Hamiltonian of this system is given by

$$\mathcal{H}_N = -\frac{J}{2N}\sum_{i,j} s_i s_j - B\sum_i^N s_i \ .$$

$J(>0)$ is the coupling constant of the arrows (elementary magnets or spins if you want) s_i and s_j. B is an external magnetic field. Note that J does not depend on the distance between s_i and s_j (mean field approximation!). Therefore the energy of a particular state of this lattice model depends on the total number of arrows pointing up and down, i.e. N_+ and N_-, only. Each combination $(N_+ \, , N_-)$ can be realized in $N!/(N_+!N_-!)$ ways (remark: treat all N's as large, i.e. use Stirling's approximation). Thus the partition function is given by

$$Q = N! \sum_{N_+,N_-} \frac{\delta\left(N - N_+ - N_-\right)}{N_+!N_-!} e^{-\beta E(N_+,N_-)} \ .$$

(a) Show that Q can be expressed as $Q = \sum_m \exp[-\beta N \psi(m)]$. Here $m = (N_+ - N_-)/N$ is the average 'magnetization' of lattice states characterized by N_+ and N_-. Write down the explicit form of $-\beta\psi\left(m; T, J, B\right)$.

(6 points)

(b) Show that the free energy per arrow (or spin), $-(\beta N)^{-1}\ln Q$, in the thermodynamic limit is given by a term $\exp[-\beta N\psi(m_o)]$ in the sum over m. Hint: consider the leading correction in a Taylor expansion of $\psi\,(m;T,J,B)$ around its minimum at m_o and determine the contribution of this correction to $-(\beta N)^{-1}\ln Q$ in the limit $N\rightarrow\infty$. Note that here you do not need the explicit form of $\psi\,(m_o;T,J,B)$.

(6 points)

(c) Write down an equation allowing to determine the equilibrium 'magnetization' m_o. For $B=0$ provide a sketch in the $(\beta J)^{-1}-m_o$-plane, which illustrates how m_o can be obtained graphically via the aforementioned (implicit) equation (Remark: you should remember this graph from problem 14). How does the result for m_o change when $B\neq 0$.

(3 points)

(d) For $B=0$ show that in the immediate vicinity of T_c, i.e. the temperature above which $m_o=0$,

$$|m_o|\propto(T_c-T)^\beta \qquad \text{for} \quad T<T_c\,.$$

Note that here β is a number (specifically a critical exponent) and not $1/(k_BT)$! Obtain the numerical value of β.

(3 points)

(e) Comparing this problem with problem 14, which is very similar (aside from $B=0$ throughout problem 14), we conclude that the present approach takes more effort. What is the extra information obtained here justifying this additional effort (remark: the answer has nothing to do with $B\neq 0$)?

(3 points)

Problem 19: Dilute gas in a gravitational field

We consider N point-like, non-interacting masses m (ideal gas) inside a cylindrical container. The axis of the cylinder is parallel to the gravitational field of Earth. The potential energy of the particles is given by mgz, i.e. the bottom of the container is at $z=0$.

Using the classical Hamiltonian $\mathcal{H} = \sum_{i=1}^{N} \left(\frac{1}{2m} \vec{p_i}^{\,2} + mgz_i \right)$, calculate the mean energy per particle, $\langle \epsilon \rangle$, via the following two methods:

(a) generalized equipartition theorem

(3 points)

(b) explicit partition function.

(3 points)

In addition

(c) find the normalized energy probability density $p(E)$ for this gas.

(6 points)